Math 121 2.6 - 1st Composition + Chain Rule

Objectives

1) Find composition of two functions (review)
2) Identify which functions were composed to obtain given expression (review)
3) Use the chain rule to find the derivative of a composition.

① Find the composition

$f(x) = \sqrt[3]{x}$

$g(x) = x^3+8$

$h(x) = x-5$

a) $f(g(x))$ d) $g(f(x))$ g) $f(g(h(x)))$ j) $f(h(g(x)))$

b) $f(h(x))$ e) $h(f(x))$ h) $g(h(f(x)))$ k) $g(f(h(x)))$

c) $g(h(x))$ f) $h(g(x))$ i) $h(f(g(x)))$ l) $h(g(f(x)))$

a) $f(g(x)) = \sqrt[3]{g(x)} = \boxed{\sqrt[3]{x^3+8}}$

b) $f(h(x)) = \sqrt[3]{h(x)} = \boxed{\sqrt[3]{x-5}}$

c) $g(h(x)) = (h(x))^3+8 = \boxed{(x-5)^3+8}$

d) $g(f(x)) = (f(x))^3+8 = (\sqrt[3]{x})^3+8 = \boxed{x+8}$

e) $h(f(x)) = f(x)-5 = \boxed{\sqrt[3]{x}-5}$

f) $h(g(x)) = g(x)-5 = x^3+8-5 = \boxed{x^3+3}$

g) $f(g(h(x))) = \sqrt[3]{\text{result c)}} = \boxed{\sqrt[3]{(x-5)^3+8}}$

h) $g(h(f(x))) = (\text{result e)})^3+8 = \boxed{(\sqrt[3]{x}-5)^3+8}$

i) $h(f(g(x))) = (\text{result a)})-5 = \boxed{\sqrt[3]{x^3+8}-5}$

j) $f(h(g(x))) = \sqrt[3]{\text{result f)}} = \boxed{\sqrt[3]{x^3+3}}$

k) $g(f(h(x))) = (\text{result b)})^3+8 = (\sqrt[3]{x-5})^3+8 = x-5+8 = \boxed{x+3}$

l) $h(g(f(x))) = (\text{result d)})-5 = x+8-5 = \boxed{x+3}$

(2) Identify which functions were composed to obtain the given expression.

a) $(5x^2 - x + 2)^4$

inside $= 5x^2 - x + 2 = f(x)$
outside $= x^4 = g(x)$
expression $= g(f(x))$

b) $\sqrt{\frac{x-1}{x+1}}$

inside $= \frac{x-1}{x+1} = f(x)$

outside $= \sqrt{x} = g(x)$

expression $= g(f(x))$

c) $\frac{\sqrt{x} - 1}{\sqrt{x} + 1}$

inside $= \sqrt{x} = f(x)$
outside $= \frac{x-1}{x+1} = g(x)$

expression $= g(f(x))$

d) $\frac{1}{\sqrt[3]{(x^2+x-9)^2}}$

$= (x^2+x-9)^{-2/3}$

inside $= x^2+x-9 = f(x)$
outside $= x^{-2/3} = g(x)$

expression $= g(f(x))$

e) $\left(\frac{1}{w^4+1}\right)^5$

inside $= \frac{1}{w^4+1} = f(w)$
outside $= w^5 = g(w)$
expression $= g(f(w))$

(3) Find the derivative two ways for $f(x) = (x^2+1)^3$

a) By simplifying first, then factor result.

b) By using the chain rule

a) $f(x) = (x^2+1)^3 = (x^2+1)(x^2+1)(x^2+1)$

$= (x^2+1)(x^4+2x^2+1)$

$= x^6 + 2x^4 + x^2$
$\quad + x^4 + 2x^2 + 1$

$= x^6 + 3x^4 + 3x^2 + 1$

$f'(x) = 6x^5 + 12x^3 + 6x$

$= 6x(x^4 + 2x^2 + 1)$

$= \boxed{6x(x^2+1)^2}$

b) $f'(x) = \frac{d}{dx}\left[\begin{array}{c}\text{outside evaluated}\\ \text{at same inside}\end{array}\right] \cdot \frac{d}{dx}\left[\text{inside}\right]$

$O(x)$ = outside $(x) = x^3$
$I(x)$ = inside $(x) = x^2+1$
$\}$ $f(x) = O(I(x))$

$O'(x) = 3x^2 \longrightarrow$ evaluate at the same inside
$O'(I(x)) = 3(x^2+1)^2$

$I'(x) = 2x \longrightarrow$ Multiply results

$f'(x) = 3(x^2+1)^2 \cdot 2x$

$= 3 \cdot 2x(x^2+1)^2$

$\boxed{f'(x) = 6x(x^2+1)^2}$

Chain Rule

$$\frac{d}{dx}(f(g(x))) = \underbrace{f'(g(x))}_{\text{compose}} \cdot g'(x)$$

multiply

Find derivatives. Fully factor results. (Do not simplify.)

④ $\frac{d}{dx}(5x^7 - 3x^2 - 2)^{23}$

power rule outside

$= \underbrace{23(\text{ same inside })^{22}}_{\text{compose}} \underset{\text{multiply}}{\cdot} (\text{ derivative of inside })$

power rule + constant multiple rule inside

$= 23(5x^7 - 3x^2 - 2)^{22} \cdot (35x^6 - 6x)$

$= 23(35x^6 - 6x)(5x^7 - 3x^2 - 2)^{22}$

$= \boxed{23x(35x^5 - 6)(5x^7 - 3x^2 - 2)^{22}}$

* Factored results will be essential in chapter 3 *

⑤ $\frac{d}{dx} \frac{1}{\sqrt{x^6 + 5x^4 - 2x^3 + 7}}$

$= \frac{d}{dx}(x^6 + 5x^4 - 2x^3 + 7)^{-1/2}$

rewrite as negative and fraction exponent ⇒ power rule outside

$= \underbrace{-\frac{1}{2}(\text{ same inside })^{-3/2}}_{\text{compose}} \underset{\text{multiply}}{\cdot} (\text{ derivative of inside })$

$= -\frac{1}{2}(x^6 + 5x^4 - 2x^3 + 7)^{-3/2} \cdot (6x^5 + 20x^3 - 6x^2)$

$= -\frac{1}{2} \cdot 2x^2(3x^3 + 10x - 3)(x^6 + 5x^4 - 2x^3 + 7)^{-3/2}$

$= \boxed{-x^2(3x^3 + 10x - 3)(x^6 + 5x^4 - 2x^3 + 7)^{-3/2}}$

<u>Generalized Power Rule</u> (is just the chain rule when outside is a power)

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

You will find references to the
<u>Generalized Power Rule</u>.

(in the book, in homework, elsewhere.)

The Generalized Power Rule is the Chain Rule for the special case when the outside function is a power x^n, but the inside function is something else, call it $g(x)$.

ex. $f(x) = (g(x))^n$

$f'(x) = n\,(g(x))^{n-1} \cdot g'(x)$

(6) $\frac{d}{dx}\left(\frac{1}{x^4+2}\right)^5$

$= \frac{d}{dx} \frac{1^5}{(x^4+2)^5}$ rewrite $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$= \frac{d}{dx} \frac{1}{(x^4+2)^5}$ simplify $1^5 = 1$

$= \frac{d}{dx} (x^4-2)^{-5}$ rewrite with negative exponents.

$= \underbrace{-5(\underbrace{x^4-2}_{\text{same inside}})^{-6}}_{\text{power rule outside}} \cdot \underbrace{(4x^3-0)}_{\text{mult by derivative of inside}}$

$= -5 \cdot 4x^3 (x^4-2)^{-6}$

$= \boxed{-20x^3 (x^4-2)^{-6}}$

$= \boxed{\frac{-20x^3}{(x^4-2)^6}}$

(7) Psychology: Learning.

After p practice sessions, a subject could perform a task $T(p) = 36(p+1)^{-1/3}$ minutes, for $0 \le p \le 10$.

Find $T'(7)$ and interpret result.

step 1: Differentiate using the chain rule

$T'(p) = 36 \cdot \left(-\frac{1}{3}\right)(p+1)^{-4/3} \cdot (1+0)$

$= -12(p+1)^{-4/3}$

Step 2: Evaluate

$T'(7) = -12(7+1)^{-4/3} = -12 \cdot \frac{1}{\sqrt[3]{8}^4} = \frac{-12}{16} = -\frac{3}{4}$

step 3: When the subject had 7 practice sessions, the number of minutes they could perform the task decreased by 3/4 min per practice session.